Can You Feel the Beat? Measuring the Quantum Beat

by

Shivam Suthendran (20499543)

A thesis presented to the University of Waterloo in partial fulfilment of requirement for PHYS 437 A

Waterloo, Ontario, Canada, 2022

ABSTRACT

TABLE OF CONTENTS

INTRO

What defines a quantum system - Indistinguishability

Why it’s important

What happens when you don’t have it

Goal of this paper

HISTORICAL and TECHNICAL BACKGROUND

Quantum Optics

Temporal Mode Functions

Coincidence Counts

Hung Ou and Mandel

Hanbury Brown and Twiss

Correlation Functions

METHODOLOGY AND RESULTS

CONCLUSIONS

What does this paper seek to do?

How it achieves this with relevant results

Possible Ramifications

Future Considerations

# CHAPTER 1

# INTRODUCTION

What makes up the quantum nature of a system? Is it particle size? Is it the inclusion of the Planck constant? Is it that the system is probabilistic? While these may contribute to the quantum nature of a system, one of the hallmarks of a quantum system is in the indistinguishability of particles. Indistinguishability, also called indiscernible or identical particles, is the consideration that one particle can not be distinguished from another. While it is not the only hallmark of a quantum system, indistinguishability is a purely quantum effect (1) and can be used to determine the nature of a system in question.

While the concept seems simple, it has profound effects, particularly in quantum optics. But before we get there, let us further elaborate on the concept of indistinguishability. In classical mechanics, while two particles may be identical in their inherent qualities, we are still able to identify one from another using their relative positions and momenta. Following this, we are then able to follow the subsequent trajectories and interactions of these particles. However, suppose we considered two electrons instead? What would the evolution of their collision look like?

|  |  |
| --- | --- |
|  |  |
| Fig 1.1 – Which is the correct path? | |

For two particles that are inherently not different from one another, which path seems more correct? Especially when taking Heisenberg’s uncertainty principle into consideration, a particle’s location ends up having very little meaning outside of the time of measurement. We now have no real way to distinguish the particles, and as such, we cannot follow particle trajectory in the same way as in the classical case.

What we have above seems rather abstract, more like a thought experiment, how could this possibly have any real-life applications? However, as mentioned above, the lack of indistinguishability can lead to some very interesting physical results. This leads us to the work done by Legero, Wilk, Kuhn, and Rempe. In 2003 they produced a paper detailing the mathematical framework that could demonstrate the distinguishability in photons. In particular, they found that if there was a difference in frequency between the photons, this would lead to an effect they called the quantum beat.

|  |
| --- |
|  |
| Fig 1.2 – The Beat Effect |

What is the quantum beat? When two photons, that are distinguishable due to a difference in frequency, and have a temporal delay with respect to one another that is shorter than their wave packet length, interact; they can create an interference effect that looks like a beat effect. This can be demonstrated experimentally by graphing coincidence counts of the output photons.

In this paper I seek to convince the reader that the presence of a quantum beat can be used to determine the distinguishability of photon sources. First, I must calculate the second order correlation function, I then take this result and integrate it over all possible detection times (τ), to determine the probability distribution of all possible detection times. This will be contrasted with the probability distribution of the Hong-Ou-Mandel experiment, as well as with Legero, Wilk, Kuhn, and Rempe’s work. However, I’m getting a bit ahead of myself, before we get there, in Chapter 2, I will provide historical and mathematical context of the experiment and the analytic techniques used in this thesis. In Chapter 3, I will provide my own contributions to the demonstration of the quantum beat, with a detailed analysis of my findings. Finally, in Chapter 4, I will make my conclusions with future considerations and improvements. It is my genuine hope that you will have learned as much as I have through the reading of this thesis.

# CHAPTER 2

# CONTEXT

Before I can delve into the work that I have done over this project, it is important that I provide context for the work, this includes historical and technical context along with an analysis of present work. This is done in three parts:

1. The Framework of Quantum Information Processing and Quantum Optics
2. The Hong Ou Mandel Experiment
3. Hanbury-Brown Twiss Experiment

## 2.1 Quantum Background

### 2.1.1 Quantum Information Processing

First proposed by Paul Benioff in 1980, and built upon by Feynman in 1981, quantum computation takes on the framework of quantum mechanics to perform simulations that are far beyond the ability of classical computation. In particular, it takes advantage of superposition and entanglement; both of which are fundamental to quantum mechanics. Let us consider the superposition principle, and its use in quantum computation. Put simply, prior to measurement, a given quantum state is said to have access to and exists in multiple states. Upon measurement, a state is picked with some probability, and no information of its previous state is accessible.

Before we continue in more depth, let us consider the basic premise behind classical computation. The basic unit in classical computation is the bit, which takes on either a 1 or 0 value. Given these bits, logic gates can be applied to them, which serve to compare and or combine the input bit values and provide an output bit in accordance with these gates and inputs.

In quantum computation, the basic unit of computation is the qubit. A qubit’s information is encoded in the following forms:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.1)  (2.1.2) |

This notation is called Dirac notation and equations 2.1.1 and 2.1.2 denote the ground and excited states, respectively. The qubit can exist as these states individually, or as a superposition of the two states. As stated previously, this means this exists as and has access to both the 0 and 1 states. This is represented in Dirac notation as:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.3) |

This is a normalised superposition of the ground and excited states. Each state has a complex coefficient (in this case ), the modulus square of which would return the probabilities of measuring each state (which in this case would be ). Thus, upon repeated measurement of the system we would be able to attain measurements for both the ground and excited states with a probability of . This elucidates the power of quantum computation over classical computation in its ability to produce all possible results over repeated iterations.

Another important reason that quantum computation supersedes classical computation is due to entanglement. We have thus far only considered single qubits, but suppose we consider multiple qubits instead. Like the single qubits, they exist as a superposition of basis states; however there exist states that can not be written as separable products of the individual states.

Consider a second qubit of the following form:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.4) |

We can write binary qubit state, the tensor product of the two states, is given as follows:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.5) |

This is a separable state, as it can be written as a product state of the two qubits. The state is spanned by four basis states, of which each qubit brings two basis states.

A non-separable state is one that can not be decomposed into a product of the two qubits. There exists some correlation term that can not be cleanly divided between the two qubits. When the state has this form, it is said to be an entangled state. The Bell states are examples of non-separable states, and they represent all combinations of maximally entangled basis vectors. They are:

|  |  |  |
| --- | --- | --- |
|  |  | (2..6) |
|  |  | (2..7) |
|  |  | (2..8) |
|  |  | (2..9) |

### 2.1.2 States and Measurement

#### 2.1.2.1 Object Property Representation

I have loosely used the terms measurement and state but have yet to provide context for either. Suppose we were to consider an electron. What makes an electron an electron? What do we seek to know from the electron? Well, we can consider the properties of an electron, such as mass, angular momentum, spin etc. These properties give us information about the ‘state’ of an electron.

Mathematically, the set of all possible states within a property, spans a space called the Hilbert space. As an example, the set of all spin states of an electron spans a Hilbert space that is two-dimensional. This space has a set of vectors, called basis vectors, that span the entirety of the Hilbert space. As per the spin example, these basis vectors are the up and down spin states. One can generate any state in this Hilbert space as a linear combination of these basis states. Thus, a specific electron spin state is a vector within this Hilbert space, and as such, can be written as a linear combination of the up and down spin states.

In Dirac notation, these vector states are represented as ‘kets.’ Using our spin example again, the spin down state looks as follows:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.10) |

We can see that this also has a column vector form. These column vectors can be transposed into row vectors, which belong to a corresponding space that is dual to the Hilbert space. These dual vectors are a set of functions that act upon our vectors and produce scalars. This is represented in Dirac notation via ‘bras,’ which look as follows:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.11) |

This the bra corresponding to equation 2.1.10. A small but important note is to remember that the Hilbert space is a complex vector space, as such dual vectors are conjugate transposes of the vectors, rather than just the transposes.

#### 2.1.2.2 Accessing these Properties

Now that we know what states are how the represent properties of an object in question, how are we supposed extract this information for any use? Much like in the classical world, information is gained via measurement. The act of measurement is represented mathematically via an operator. Each basis vector in the Hilbert space is an eigenvector that has an associated eigenvalue. When an operator acts upon these vectors, it pulls out the associated eigenvalue. Again, using the spin analogy, suppose I wanted to measure the spin in the z direction:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.12) |

First the operator acts on the ket to pull out the appropriate eigenvalue, . The set of all basis vectors form an orthonormal set, and when taking their inner product, we end up with a 0 if they are different, and a 1 if they are the same. In this case they are and thus we are left with our measurement:

In a superposition this will pull out a weighted measurement of the mean, suppose we have some state , which looks as follows:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.13) |

Then taking the same measurement of the spin as in equation 2.1.12, gives us:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.14) |

This is also called the expectation value of the system, and gives the weighted average measurement of an operator. Often the averages of a given operator are denoted as , wherein is the operator in question.

### 2.1.3 Quantum Harmonic Oscillator

I have used the ground and excited states quite liberally in the section 2.1.1; however, I have not defined the framework from which they come from. Much like in classical computation, quantum computation also utilises a two-level system. One way this occurs is by piggybacking off of the solutions for the quantum harmonic oscillator. We define this problem as follows:

|  |
| --- |
|  |
| Fig 2.1.1: Particle on a Spring |

Consider a typical problem in classical mechanics: a particle oscillating on a spring. We know that this has the following Hamiltonian:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.15) |

Wherein:

* is the particle momentum
* is the particle mass
* is the particle position
* And is the particle oscillation frequency

In order to make this a ‘quantum’ harmonic oscillator, I will quantize the variables. For the purposes of this paper, this will seem as though I’m simply putting hats on the variables while holding the canonical commutation relationship []= in mind. However, the mathematics is far more in depth and beyond the scope of this paper.

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.6) |

The typical approach is to solve this as a second order ordinary differential equation which will yield the following wavefunction:

|  |  |  |
| --- | --- | --- |
|  | ) | (2.1.17) |

Wherein:

* are the Hermite Polynomials and are defined as follows:

|  |  |  |
| --- | --- | --- |
|  | ) | (2.1.18) |

* defines the energy levels of the system

However, this is not quite what we’re looking for. In order to get a more suggestive form, we can consider a small redefinition of the variables, which was first conceived by Paul Dirac:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.19) |
|  |  | (2.1.20) |

These are called ladder operators and represent the action of lower or raising the energy level or the state. As an example, the raising operator can act on the ground state to given us the first excited state:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.21) |

Similarly, the lowering operator can take this excited state and lower it as follows:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.22) |

One can take these operators and rewrite the Hamiltonian as follows:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.23) |

Wherein the product of the ladder operators can be recast as , the number operator. This counts the number of excitations. For example, if the number operator were to act on the first excited state, it would count a single excitation:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.24) |

### 2.1.4 Quantum Optics

Of the multiple proposed modes for computation, one of the first and most popular is the photon. Information is encoded into the photon in multiple possible modes: frequency, spatial mode, polarisation, and time of arrival. This is a natural choice for candidacy due to the photon’s high durability (weak interactions with the environment) and high mobility. In this thesis, we will pay particular attention to the frequency mode of encoding. The harmonic oscillator framework is a natural fit for describing photon interactions such as excitation, absorption, and photon counting.

#### 2.1.4.1 Quantization of an Electromagnetic Field

However, we can not employ the harmonic oscillator convention as is. First, I must demonstrate that this framework is indeed a natural fit for the photon. To do this first consider the classical electromagnetic fields governed by Maxwell equations (in a vacuum):

|  |  |  |
| --- | --- | --- |
|  |  | (2..25) |
|  |  | (2..26) |
|  |  | (2..27) |
|  |  | (2..28) |

Wherein:

* is the nabla/del operator
* is the electric field
* is the magnetic field
* is the speed of light within a vacuum

We can take the curl of the last two equations (equations 2.1.27 and 2.1.28), and when we do we find the following second order differential equations:

|  |  |  |
| --- | --- | --- |
|  |  | (2..29) |
|  |  | (2..30) |

The solutions for the electric field have the following form:

|  |  |  |
| --- | --- | --- |
|  |  | (2..31) |

Wherein:

* is a vector that denotes an arbitrary direction
* is the wave vector in the direction of wave propagation
* denotes the polarization of the vector, perpendicular the direction of propagation
* is the frequency of the wave, such that

|  |  |  |
| --- | --- | --- |
|  |  | (2..32) |

* are spatial plane wave solutions of the form (also called mode functions):

|  |  |  |
| --- | --- | --- |
|  |  | (2..33) |

* + With representing the electric field component with respect to polarization
  + representing the unit volume within which the field is confined
* and representing the complex amplitudes associated to each temporal mode

As these solutions are found with the assumption that the wave is propagating in a vacuum, there are no free charges. As such this implies that the polarization is orthogonal to the direction propagation.

Now to consider the magnetic field, which can be found using equation (2.1.21)

|  |  |  |
| --- | --- | --- |
|  |  | (2..34) |

Wherein:

* are spatial plane wave solutions of the form (also called mode functions):

|  |  |  |
| --- | --- | --- |
|  |  | (2..35) |

* + With representing the magnetic field component with respect to polarization and given by the following relationship:

|  |  |  |
| --- | --- | --- |
|  |  | (2..36) |

Given the electric and magnetic field equations define above, we can now determine the energy of a classical electromagnetic field as follows:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.37) |
|  |  | (2..38) |

These look very close to quantum harmonic oscillator solutions, however there are some considerations before we can call them similar. As with the quantum harmonic oscillator, we must quantize the complex amplitudes. Again, this is more complex than what it seems (simply putting hats atop the operators and changing the star to a dagger); however, suffice it to understand that they are more complex than what is given. We must also consider that these operators obey bosonic commutation relations, ie:

|  |  |  |
| --- | --- | --- |
|  |  | (2..39) |

Our Hamiltonian then becomes:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.40) |
|  |  | (2.1.41) |

Which takes advantage of the bosonic commutation relationships to get the final equality. We can safely conclude that the photon indeed follows a harmonic oscillator framework, with the careful consideration that ladder operators follow a bosonic commutation relationship.

#### 2.1.4.2 Temporal Mode Functions

The electromagnetic wave solutions are found with consideration to polarization, but as I mentioned before, the photon can encode in multiple modes. In this thesis we pay particular consideration to the frequency mode of encoding. As such, I can not use the polarization mode functions, and need to seek solutions associated to frequency instead. How do we go about this?

Mode functions were first characterized by Glauber and Titulær in 1966 and were proposed as a solution to the real-world problem that photon emissions are never exactly monochromatic, but instead a wave packets that hold multiple frequencies. They considered that the spectral width, the spectrum of frequencies contained by a photon, is roughly the inverse of the duration of the wave packet. This gave rise to the idea of a spatio-temporal wave packet, that was not characterized by polarization. As time and frequency are Fourier transforms of one another, they were then able to characterize the wave packet with respect to its frequency as well. This derivation will be done entirely without Fourier transforms, and as such will not be defined further in this thesis. Suffice it to say that there is an intrinsic relationship between time and frequency that allows for this transformation to occur.

We shall pay particular attention to defining temporal mode functions. Much like equation 2.1.31, we will assume a similar general form for the equation of an electric field (in the frequency domain) as follows:

|  |  |  |
| --- | --- | --- |
|  |  | (2..42) |

We seek to create new creation and annihilation operators that will generate discrete photons which contain a collection of frequencies. Thus, we define a linear superposition of these and which are modified by a set of complete and orthogonal weight functions . This looks as follows:

|  |  |  |
| --- | --- | --- |
|  |  | (2..43) |
|  |  | (2..44) |

These operators obey the bosonic commutation relationships:

|  |  |  |
| --- | --- | --- |
|  |  | (2..45) |

As mentioned above, these orthogonal weight functions, , are also complete. This means that they obey the following relationship:

|  |  |  |
| --- | --- | --- |
|  |  | (2..46) |

Using this, we can now define inverse relationships of the following form:

|  |  |  |
| --- | --- | --- |
|  |  | (2..47) |
|  |  | (2..48) |

We now have all the ingredients to create our temporal mode functions. These are defined using equations (2.1.34, 2.1.39, and 2.1.40) as:

|  |  |  |
| --- | --- | --- |
|  |  | (2..49) |
|  |  | (2..50) |

These temporal mode functions are key to the set up of the quantum beat problem, as they play the role of defining the ‘shape’ of the photon.

This thesis will use two types of temporal mode functions for photons:

The Gaussian Photon:

Which has the following mode function:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.51) |

Wherein:

* denotes the pulse width
* represents the temporal offset between the pulses

representing photon a

* + representing photon b
* representing the difference between the frequencies
* Frequencies that are centered about
  + Thus representing photon a
  + And representing photon

The Gaussian mode functions defines an idealized wave packet. It is called a ‘packet’ as it consists of multiple wavenumbers clustered about a single value. This mathematically exemplifies ideal monochromatic emission.

The Lorentzian Photon:

Which has the following mode function:

|  |  |  |
| --- | --- | --- |
|  |  | (2.1.52) |

Wherein:

* denotes the decay rate of the pulse
* represents the temporal offset between the pulses
  + representing photon a
  + representing photon b
* representing the difference between the frequencies
* Frequencies that are centered about
  + Thus representing photon a
  + And representing photon
* denotes the unit step function

This shape arises in the case of homogenous broadening. In general, photon emissions are due to excitation and subsequent de-excitation of a quantum system. Each energy level of this quantum system has its own energy profile that is different from one energy level to the next. This profile is proportionate to the emission spectrum of the emitted photon upon de-excitation. The energy level is also proportionate to the natural lifetime of the excited state in question. The lifetime of the excited quantum system is not an exact model but exists upon a distribution about an average lifetime. We can also then infer that the energies, which are proportionate to lifetime also exist on a distribution and as such are broadened. The contribution of these effects on the photon are called homogenous broadening, and are better demonstrated via a Lorentzian curve, rather than a Gaussian.

~

In this section we have successfully defined the photon with respect to frequency under a quantum framework. We are now able to employ them for experimentation which will be further elaborated upon in the next section – The Hong, Ou, Mandel Experiment

## 2.2 Hong, Ou, and Mandel Experiment

A landmark experiment in the field of quantum optics, the Hung-Ou-Mandel (HOM) effect was first demonstrated by Chung Ki Hong (홍정기), Zheyu Ou (区泽宇), and Leonard Mandel in 1987 at the University of Rochester. In this section we will dissect this experiment and put it under a mathematical framework that will make it accessible for our experimentation. We will begin with the experimental set up with a follow up discussion on the results and following analysis.

### 2.2.1 Experimental Set Up

Though this experiment is a hallmark in quantum optics, its execution is deceptively simple. It employs the use of a beam splitter upon which two photons are incident, the resultant photons and their detections are then analyzed. But I get ahead of myself, let me break this down a little further, starting with the beam splitter.

The beam splitter is an optical device that takes any input beam and divides it into two ‘arms.’ In most experiments, a beam splitter splits the beam into two constituent halves of the input beam, and as such is called a beam splitter. However, there are other beam splitters, such as the beam splitter, which splits the beam into and output arms. The beam splitter is generally a cube of glass which is made up of two glass prisms.

* + - As two photons are incident upon a beam splitter, there are 4 possible outcomes
      * Both photons are reflected
      * Both photons are transmitted
      * One photon is transmitted and the other reflected (Multiplicity of 2)
        + If photons are purely indistinguishable then it would be impossible to determine if the upper or lower photon gets transmitted
* Mathematical Framework

2.2.2 Results

* Hallmark effect of quantum mechanics
  + Two photodetectors are placed in the output modes of the beam splitter
  + A coincidence count is measured as a photon is incident upon both photodetectors
  + As time passes and the photons overlap interfere perfectly, i.e. there is no mode in which they are distinguishable from one another, the coincidence counts drop to 0
  + When this occurs, this is considered the experimental signature, and is shown as a dip in the coincidence counts as shown below.
* Mathematical result

This experiment provides us with the experimental setup to measure the quantum beat.

2.3 Hanbury-Brown Twiss Experiment

* Rather than quantum optics, we are going to take a slight detour to astronomy, taking note of the work of Hanbury Brown and Twiss
* Developed an interferometer that was an improvement on the Michelson stellar interferometer
* Given a Michelson interferometer, light from a star is collected by two mirrors which are separated by a distance
* If the light from the source has the same frequency, then an interference pattern forms on the focal plane, however this is not the case, the intensities will simply add.
* Upon multiple variations in d, and thus in the interference patterns, one can use this information to determine the angular size of the star to be measured.
* The mirrors used in this interferometer are relatively small and as the distance between them becomes too large, the angular resolution becomes compromised
* Instead of two mirrors, Hanbury- Brown and Twiss instead used a beam splitter. In order to eliminate the issue of losing angular resolution due to large distances
* This beam splitter will act to split the light incident upon into two output ports.
* The number of pulses output on the photodetectors are recorded along with temporal separation between outputs upon the photodetectors
* Having this, what do we do with this?

Correlation Function

* Consider a beam that is incident upon a beam splitter
* At the transmitted and reflected arms of the beam splitter, there is a photo multiplier at the end
  + Serves to amplify the current to a measurable electrical signal
* The signals were connected with a unit that multiplied and averaged the signals
* As such, the result was proportional to, the time average of the two signals
* The output time average of these signals are proportional to the light intensities prior to being incident upon the photomultipliers
* In order to glean information from these results, we now consider the second order correlation function:
* Counts are proportional to number of photons incident upon the photomultiplier, therefore instead we can rewrite the second order correlation function can be rewritten as:

This is the last piece of the puzzle, the mode of analysis used to analyse the results of the experiment.

Now that I’ve established my building blocks, in the next session I wish to use them in such a way that I can put them together to measure the quantum beat.

# CHAPTER 3

# METHODOLOGY AND RESULTS

Legero, Wilk, Kuhn, and Rempe demonstrated the mathematical background for the two-photon effect and demonstrated the quantum beat by graphing the two-photon coincidence probability against the detection time. In this thesis I seek to demonstrate the same effect via the second order correlation function.

Consider again the Hong-Ou-Mandel experiment with two photons in put on a beam splitter, the second order correlation function associated to the two output photons are as follows:

Wherein the number operators can be recast as the product of adag and a, this then becomes:

This uses the normal ordering convention that keeps the creation operators to the left and the annihilation operators to the left.

The initial state of the system corresponds to the following quantum state:

Wherein the the creation operators of photon one and two are applied on some ground state. Thus, the output creation operators can be written as a normalized linear combination of the input creation operators. By doing this we can rewrite equation () with respect to the creation and annihilation operators of the first and second photon, which are as follows:

Having this we can test different input photons to and determine their correlation functions. We will do this by first considering two idealized Gaussian photons, then a more realistic Lorentzian photon, and finally a stream of Lorentzian photons.

Case 1: The Gaussian Photon

Consider a Gaussian photon which has the following form:

Its correlation function is then:

Suppose we consider the simple case where there is no difference in frequency, ie delta=0 and. We then find the experimental signature of the HOM experiment. That is to say that if there is no delay between the photons (ie. Delta tau=0), then our second order cross correlation function suggesting that the light is antibunched.

We can also appropriately change the delay between photons and we still find that no our second order cross correlation function still suggests that the light is antibunched. This also suggests that any interference is quantum in nature (as antibunching is a uniquely quantum phenomenon).

Suppose instead we include a frequency difference, which makes these photons distinguishable. We see that there are still no coincident detections (tau=0), however, the second order correlation function oscillates with respect to detection time difference tau.

# CHAPTER 4

# CONCLUSIONS AND FUTURE CONSIDERATIONRS

It’s the final brain cell